SOLUTIONS OF PROBLEMS.

430 (Algebra) [March, 1915]. Proposed by V. M. SPUNAR, Chicago, Illinois.

Solve the following equations algebraically and graphically:

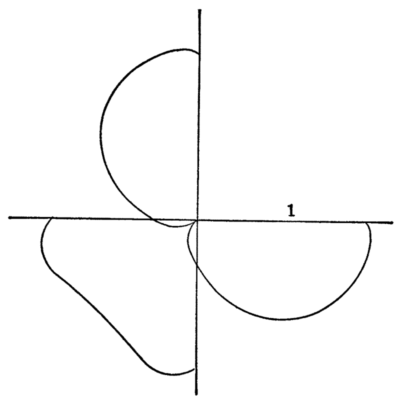
$$x^y + y^x = xy, \qquad x^x + y^y = x + y.$$

SOLUTION BY A. A. BENNETT, University of Texas.

There is no pair of real values, with the doubtful exception of (0, 0), which furnishes a simultaneous solution to the two equations given.

The equations being transcendental no "algebraic" solution is attempted. The discussion here is analytical, supplemented by a graph.

In examining the second of the above equations, it is convenient to distinguish three types of real values, viz., (*A*), the type in which the variable has any positive real value; in both (*B*) and (*C*) are considered only negative and rational numbers when written as irreducible fractions; in (*B*) the numerator of such fractions is even, while in (*C*) it is odd. Irrational negative values or rational negative values which when reduced to lowest terms have even denominators cannot appear among real solutions.



The following graph is liable to misinterpretation and will be qualified.

In the first quadrant there is a single point, (1, 1).

In the second quadrant, the graph is discontinuous, x being of type (*C*) and y of type (*A*). An infinite number of points certainly exist satisfying the equation.

In the fourth quadrant, x is of type (*A*) and y of type (*C*). Above remarks apply by symmetry.

In the third quadrant there are three branches to consider: (i) an apparent continuation of the branch of the second quadrant, (ii) an apparent continuation of the branch of the fourth quadrant, (iii) a separated branch containing $(-1, -1)$. Untractable questions of irrationality make doubtful the existence of any actual solutions in the third quadrant other than $(-1, -1)$. If any exist, they are on the branches above noted and are of the following types. On (i) x is of type (*C*) and y of type (*B*), on (ii) x is of type (*B*), and y of type (*C*); on (iii) x and y are each of type (*C*).

A real evaluation of $f(x, y) \equiv x^y + y^x - xy$ is only possible for both x and y rational, so far as points of the above graph are concerned. A study of $f(x, y)$ on the points of the graph shows it, even if existent, very far from zero, except as the origin is approached on branches (i) and (ii) in the third quadrant.

519 (Geometry) [September, 1917]. Proposed by OTTO DUNKEL, Washington University.

Given the conjugate axes $A'OA$ and $B'OB$ of an ellipse, points of the curve may be constructed as follows: Drop the perpendicular BM' to OA and produce it to N' so that $BN' = AO$. Draw a straight line through O and N'. Upon a straight edge, say that of a slip of paper, the points N, M, and P are marked so that $NM = N'M'$ and $MP = M'B$. Place the straight edge so that N falls on ON' and M on OM' and mark the position of P. This gives a point of the ellipse and by sliding the straight edge into new positions other points may be rapidly obtained. If the axes are perpendicular this gives the familiar trammel construction. Prove the correctness of this construction.

I. SOLUTION BY F. H. SAFFORD, University of Pennsylvania.

Let the axes of coördinates be taken along OA and OB, calling OA and OB, a and b, respectively. Then for P, x is OK and y is KP. From the given conditions, $MP = M'B = b \sin \omega$; hence, from the triangle MKP, (if $\alpha = \angle KMP$),